THE MOTION OF TURBULENT LIQUID FILMS

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The results of theoretical investigations into the motion of turbulent liquid films are set out. Problems of stabilizing the film in the initial part of a tube and of the motion of liquid and gas in the rod mode are considered, and the possible conditions for the formation of gas locks are elucidated.

In the tubes of vertical evaporators such as those used in the food industry, both thin laminar and thick turbulent liquid films may flow. In the latter case the term "film" is rather arbitrary. A number of research workers have studied the motion of laminar liquid films [1-4]; turbulent films were considered in [5]. In this paper we shall set out the latest results concerning the motion of turbulent liquid films in vertical tubes.

Stabilization of the Motion of the Film in the Initial Part of the Tube. Let liquid and gas pass into the tube in the manner indicated in Fig. 1. Clearly, after contact between the components, the stabilization of their motion will be interlinked. Let us assume that the motion of the components at the entrance is axially symmetrical, the liquid and gas are incompressible, the mass forces acting on the gas may be neglected, the liquid has a laminar underlayer (the constant thickness and the velocity field of which are determined as in [5]), and the film thickness varies insignificantly. Subject to these assumptions, the steady-state motion of each component outside the laminar underlayer along the longitudinal axis x is described by the equation

$$\rho_l \frac{\partial v_l^2}{\partial x} = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(\rho_l r \nu_l^T \frac{\partial v_l}{\partial r} \right) - \frac{\partial P_l}{\partial x} + (\rho_l - \rho_1) g + \Phi_l, \tag{1}$$

where the function Φ is determined from the equation of motion taken in projection along the r axis in the following way:

$$\Phi_l = -\frac{1}{r} \cdot \frac{\partial}{\partial r} (\rho_l v_{rl} v_l) = -\frac{1}{r} \cdot \frac{\partial}{\partial r} \int r \left[\frac{\partial P_l}{\partial r} + T_{rxl} + \frac{1}{r} \cdot \frac{\partial}{\partial r} (r \rho_l v_{rl} v_l) \right] dx.$$

Following [5-8], in Eq. (1) we may make the substitutions

$$\nu_l^T = a_l^2 L_l (v_l - v_{\text{lam}}), \tag{2}$$

$$P_2 = P_1 + \frac{\sigma}{R_2 - \delta} \approx P_1. \tag{3}$$

The exact estimation of the dimensions L_l is at the moment quite a difficult problem. Remembering that in the case under consideration the motion of the components takes place quite close to the entrance into the tube, we may to a fair accuracy consider that for the liquid $L_2 \approx \delta$ while for the gas $L_1 \approx R_2$.

After making the corresponding substitutions, we refer the equation to the liquid component by putting $l = 2$. The integral of the resultant equation should obey the following boundary and initial conditions

$$\begin{aligned} v_2 &= v_{2\text{л}} \quad \text{for} \quad r = R_2 - \delta_{\text{lam}},\ 0 < x < \infty, \\ v_2 &= v_{20} \quad \text{for} \quad R_2 - \delta < r < R_2,\ x = 0. \end{aligned} \tag{4}$$

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The general expression derived by the Fourier method and satisfying (4) takes the following form for a sufficiently thin laminar underlayer

$$u_2=(v_2-v_{2\,\mathrm{lam}})^2=\sum_{n=0}^{\infty}\left[c_{2n}\exp\left(-\int_0^x a_2^2 y_{2n}^2\frac{\delta}{R_2^2}\,dx\right)\right.$$

$$\left.+\exp\int_0^x a_2^2\delta A_{2n}dx\int_0^x\exp\left(-\int_0^x a_2^2\delta A_{2n}dx\right)\left(\frac{df_{2n}}{dx}+b_{2n}\right)dx\right]Z_0\left(y_{2n}\frac{r}{R_2}\right). \tag{5}$$

In Eq. (5) we have introduced the following nomenclature:

$$\left.\begin{matrix}c_{2n}\\ \dfrac{df_{2n}}{dx}\\ b_{2n}\end{matrix}\right\}=\frac{2}{R_2Z_1^2(y_{2n})}\int_0^1\left\{\begin{matrix}v_{20}^2\\ -\dfrac{1}{\rho_2}\dfrac{\partial P}{\partial x}+\Phi_2\\ \left(1-\dfrac{\rho_1}{\rho_2}\right)g\end{matrix}\right\}rZ_0\left(y_{2n}\frac{r}{R_2}\right)dr; \tag{6}$$

Z_ν are Bessel functions of the first kind and the ν-th order:

$$A_{2n}=\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}Z_0\left(y_{2n}\frac{r}{R_2}\right)+\frac{\partial^2}{\partial r^2}Z_0\left(y_{2n}\frac{r}{R_2}\right)\right]Z_0^{-1}\left(y_{2n}\frac{r}{R_2}\right); \tag{7}$$

y_{2n} are the positive roots of the equation;

$$Z_0(\lambda r)|_{r=R_2}=0\quad\text{for}\quad 0<\lambda<\infty. \tag{8}$$

The integral (5) includes the as-yet unknown coefficients f_{2n}, depending not only on the distribution of the pressure and the field of transverse velocities of the liquid but also on the kinematics of the longitudinal velocities of the gas component. Let us therefore turn to the derivation of a formula giving the gas velocities. To this end we make use of Eq. (1), referred to the gas component, with due allowance for (2) and (3) and with $v_{1\mathrm{lam}}=0$, i.e., we consider that, at the entrance, the laminar underlayer in the gas component on the "boiling" surface of the liquid is extremely insignificant. The integral of this equation has to satisfy the conditions

$$\begin{aligned}&v_1=v_{10}\quad\text{for}\quad 0<r<R_2-\delta,\quad x=0;\\ &v_1=v_{2\delta}\quad\text{for}\quad r=R_2-\delta,\quad 0<x<\infty.\end{aligned} \tag{9}$$

If we introduce the nomenclature $u=2v_{2\delta}\Delta v_1+\Delta v_1^2$, where Δv_1 is the gas velocity in excess of $v_{2\delta}$, the solution takes the form

$$u=\sum_{n=0}^{\infty}\left\{c_{1n}\exp\left[-\int_0^x a_1^2y_{1n}^2\frac{R_2}{(R_2-\delta)^2}dx\right]+\exp\left(\int_0^x a_1^2R_2A_{1n}dx\right)\int_0^x\exp\left(-\int_0^x a_1^2R_2A_{1n}dx\right)\right.$$

$$\left.\times\left(\frac{df_{1n}}{dx}+b_{1n}\right)dx\right\}Z_0\left(y_{1n}\frac{r}{R_2-\delta}\right), \tag{10}$$

where the coefficients c_{1n}, f_{1n}, and b_{1n} are determined by means of (6), with y_{2n} replaced by y_{1n} and correspondingly $v_{10}^2, -1/\rho_1\cdot\partial P/\partial x+\Phi_1$. The values of y_{1n} are defined as the positive roots of

$$Z_0(\lambda r)|_{r=R-\delta}=0\quad\text{for}\quad 0<\lambda<\infty. \tag{11}$$

The function A_{1n} is found by means of Eq. (7) with

$$Z_0=Z_0\left(y_{1n}\frac{r}{R_2-\delta}\right).$$

The coefficients c_{2n} and c_{1n} in integrals (5) and (10) are determined simply by the velocity distribution of the components at the entrance into the tube. The development of flow along the length of the tube is determined by the coefficients f_{2n}, f_{1n}, b_{2n}, and b_{1n}. The latter establish a mutual relationship between f_{2n} and f_{1n}, satisfying the condition of constant volumetric rates of flow of the components

$$Q_2=2\pi\int_{R_2-\delta}^{R_2}v_2rdr;\quad Q_1=2\pi\int_0^{R_2-\delta}v_1rdr. \tag{12}$$

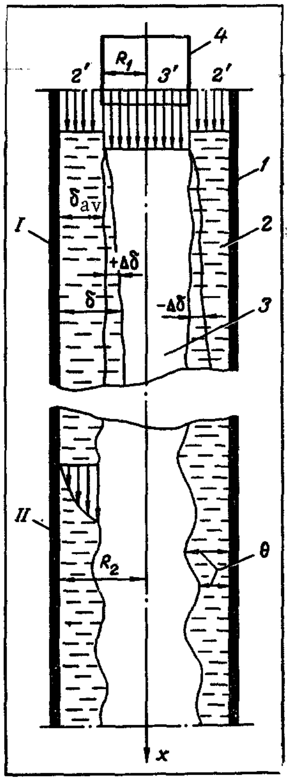

Fig. 1. Scheme of flow: I) stabilization section; II) section of steady motion; 1) tube; 2) liquid; 3) gas; 2', 3') velocity profiles of the liquid and gas at the entrance into the tube; 4) cylindrical insertion piece.

The coefficients f_{2n} in turn also have to satisfy the condition of continuity of the tangential stresses at the interface between the components in any cross section of the two-component flow

$$\rho_1\nu_1 \frac{\partial}{\partial r} v_1 \Big|_{r=R_2-\delta} = \rho_2\nu_2 \frac{\partial}{\partial r} v_2 \Big|_{r=R_2-\delta}. \tag{13}$$

Generally speaking, solutions (5) and (10) are only valid for a short entrance section, since on moving away from the entrance the effect of the walls increases by ν_l^T. In order to refine the solution, we must combine (5) and (10) with the conditions of flow in the steady section by refining the indices $\int a_l^2 A_{ln} dx$.

Stabilization of the combined motion of the components takes place at a fair distance from the tube entrance, equal to (10-12)d. Let us now study this state of affairs.

Rod-Like Motion of the Components. In considering this mode, we shall attempt to describe the motion of each component in both the turbulent core and the laminar underlayer. In Eq. (1) we therefore introduce ν_l^T in the following form:

$$\nu_l^T = 2a_l^2 L_l v_l + \nu_l. \tag{14}$$

In the present case we shall consider that the effect of the conditions of entry on the flow are extremely insignificant, and that the influence of the wall predominates. Basing our considerations on the hypothesis of A. M. Kolmogorov [8], we therefore take the coefficient of turbulent momentum transfer as proportional to the local velocity and the distance from the wall (for the liquid) or from the interface between the components (for the gas), i.e., we assume that $L_2 = R_2 - r$ and $L_1 = R_2 - \delta - r$. Then, allowing for (14) with $\partial v_l/\partial x = 0$, in the case of uniform flow the solution of Eq. (1) relating to the gas, for example, should obey the boundary conditions

$$v_1 = v_\delta + v = v_\delta \quad \text{for} \quad r = R_2 - \delta, \tag{15}$$

where v is the gas velocity in excess of v_δ. If we introduce the nomenclature

$$\eta = 1 - \frac{r}{R_2 - \delta}, \quad \varphi = \frac{v}{v_*} a_1, \quad v_*^2 = \frac{\tau_0}{\rho_1} \text{ and } \delta_1 = \frac{\nu_1 \sqrt{\pi}}{2a_1 v_*},$$

after some simple transformations Eq. (1) may be expressed as follows:

$$\frac{d\eta}{d\varphi} - \eta \frac{d\eta}{d\varphi} = 2\eta\varphi + \frac{\nu_1}{a v_*}. \tag{16}$$

Multiplying (16) by $e^{-\eta}$ and introducing the nomenclature $z = \eta e^{-\eta}$, we convert it to the following form:

$$\frac{dz}{d\varphi} - 2\varphi z = \frac{\nu_1}{a v_*} e^{-\eta}. \tag{17}$$

The integral of the latter equation is represented by the following expression:

$$\eta = e^{\varphi^2+\eta}\delta_1 \int_0^\eta e^{-\eta} \frac{\partial}{\partial \eta} \operatorname{erf}\varphi \, d\eta, \tag{18}$$

where $\operatorname{erf}\varphi = \frac{2}{\sqrt{\pi}}\int_0^\varphi e^{-\varphi^2} d\varphi$ is the probability integral. If we introduce the function

$$f_\tau = \frac{2}{\sqrt{\pi}\,\mathrm{erf}\,\varphi} \int_0^\eta \frac{\eta e^{-\varphi^2}}{\rho_1 \nu_1 \frac{dv}{d\eta}}\, d\eta, \tag{19}$$

the formula for the excess velocity profile may be expressed in a form more convenient for practical use

$$\eta = \delta_1 (1 + f_\tau)\,\mathrm{erf}\,\varphi e^{\varphi^2}. \tag{20}$$

By comparison with other logarithmic formulas, Eq. (20) has the advantage that at a solid wall it gives zero velocities, and furthermore it directly indicates that

$$\left.\frac{dv}{d\eta}\right|_{\eta\to 0} = \frac{v_*^2}{\nu_1}, \tag{21}$$

which corresponds to the physics of the phenomenon. Experimental verification of a formula of the (18) type has been carried out on a number of occasions [6, 9, etc.].

Generally speaking, we may also derive a formula analogous to (17) for the velocity profile in a liquid film. However, considering that the thickness of the film δ is comparatively slight, we may, in agreement with [3], regard it as plane to a first approximation. On this assumption, we may write Eq. (1) for the flow under consideration thus:

$$(\rho_2 - \rho_1) g - \frac{\partial P}{\partial x} + \rho_2 \frac{\partial}{\partial \varepsilon} (\nu_2 + \nu_2^T) \frac{\partial v_2}{\partial \varepsilon} = 0 \tag{22}$$

$$(\varepsilon = \delta - R_2 + r).$$

We require that the solution of this equation should obey the boundary conditions

$$\frac{\partial v_2}{\partial \varepsilon} = \frac{\psi \tau}{\rho_2 \nu_2} \text{ and } v_2 = 0 \text{ for } r = R_2. \tag{23}$$

We seek the solution of (22) in the form of a series. Following [10], we find for the rod-like mode

$$v_2 = \frac{D\delta^2}{2\psi}\left[1 - \frac{n-\psi}{n-1}\varepsilon_1^2 + \frac{1-\psi}{n-1}\varepsilon_1^{2n}\right] - \frac{\tau_0 \delta}{\rho_2 \nu_2 \psi}\left[1 + \frac{2n+1-\psi}{2n}\varepsilon_1 + \frac{\psi - 1}{2n}\varepsilon_1^{2n}\right]. \tag{24}$$

Correspondingly, for determining the generalized friction we obtain the formula

$$\tau = \rho_2 \nu_2 \left\{\frac{dv_2}{d\varepsilon_1} + \frac{D\delta}{2\psi} \cdot \frac{n(\psi - 1)}{n-1} \varepsilon_1 [1 - \varepsilon_1^{2(n-1)}]\right\} - \frac{\tau_0}{\psi} \cdot \frac{(2n+1)(1-\psi)}{2n} [1 - \varepsilon_1^{2n-1}], \tag{25}$$

where $D = 1/\nu_2 (1 - \rho_1/\rho_2) g - (1/\rho_2 \nu_2)(\partial P/\partial x)$ and n are constants for the specified mode, $\varepsilon_1 = \varepsilon/\delta$.

As we should expect, for $\psi = 1$ the turbulent friction is equal to zero, while the velocity profile v_2 becomes parabolic. If for the laminar mode we make the substitution $\varepsilon_1 = 1 - y/\delta$ in (24), we immediately obtain the formula derived by P. Semenov [1]. For turbulent flow of the film $\psi \neq 1$, while $n \sim 10$. Since $\varepsilon_1 < 1$, even for a small distance from the wall of the tube $\varepsilon_1^{2n} \sim 0$. Hence (24) has all the properties of the v_2 profiles in a liquid film indicated by Semenov. The missing coefficients ψ and n may be determined from the known values of τ_0 and $v_{2\delta}$.

Dynamics of the Waves on the Surface of a Turbulent Film. For "thick" films the wave may have dimensions sufficient to initiate the formation of "locks." The conditions required for the formation of waves and the parameters of the latter may be derived by the method of small perturbations. Here we must remember that the generalized expression for friction in turbulent flows is variable [11]. For example, in the case of a film Eqs. (24) and (25) yield

$$\nu^T = \frac{\tau}{\frac{dv}{d\varepsilon_1}} = A + By + Cy^2 + \dots \neq \text{const.} \tag{26}$$

Let us once again assume that the radius of the tube R_2 is so large that the flow in the film may, to a first approximation, be regarded as planar. In order to describe the wave motion of the flow under consideration in the well-known manner [13, 14], we obtain the differential equation

$$\frac{\partial}{\partial t}\Delta\xi + v_2\frac{\partial}{\partial x}\Delta\xi - \frac{\partial\xi}{\partial x}\cdot\frac{\partial^2 v_2}{\partial y^2} = \nu^T\Delta\xi + 2\frac{\partial\nu^T}{\partial y}\cdot\frac{\partial}{\partial y}\Delta\xi, \tag{27}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\partial\nu^T/\partial x \sim 0$, while the current function ξ satisfies the relations

$$v_x' = \frac{\partial\xi}{\partial y}; \quad v_y' = -\frac{\partial\xi}{\partial x}. \tag{28}$$

In Eq. (28) the primes respectively indicate the horizontal and vertical velocities of the liquid due to the waves.

Let

$$\Delta\xi = \theta e^{-\lambda t + i\alpha x}. \tag{29}$$

It is clear that stable gas locks may exist with stable waves, i.e., waves neither increasing nor decreasing in time. This is quite possible if in (29) the values of λ are purely imaginary. We then have to find the conditions under which the real part of λ equals zero, i.e., we have an eigenvalue problem. Remembering that the solution of this kind of problem involves serious mathematical difficulties [14, 13], following [14], we shall confine attention to the consideration of a layer of liquid in which the profile of the velocities v_2 may be approximated by a linear relationship of the form

$$v_2 = M + Ny. \tag{30}$$

If we introduce the nomenclature

$$a_1 = \frac{A}{B}, \quad c_1 = \frac{\lambda - i\alpha M - \alpha^2 A}{B}, \quad b_1 = \frac{\alpha(iN + \alpha B)}{B},$$

$$x = (y + a_1)\sqrt{b_1} \text{ and } \theta = e^{-\frac{x}{2}}\beta(x),$$

and allow for (24), (29), and (30), Eq. (27) assumes the form

$$x\beta'' + (1 - x)\beta' - \frac{b_1 a_1 - c_1}{4\sqrt{b_1}}\beta = 0. \tag{31}$$

On the basis of (31), the values of β bounded in the range $0 < x < \infty$ are expressed by Laguerre polynomials P(L) [12]. The eigenvalues of the parameter of the equation will be

$$-\frac{b_1 a_1 - c_1}{4\sqrt{b_1}} = m \quad (m = 0, 1, 2, 3, \ldots). \tag{32}$$

Let us assume that, in the flow under consideration, a mode of motion of the components is realized for which $m = 0$. It is then not difficult to see that

$$c_x = \frac{i\lambda}{i\alpha} = \frac{N}{BM} - \frac{2i\alpha A}{M}. \tag{33}$$

It is clear that the values of λ will be purely imaginary if

$$c_x = \sqrt{\rho_*}\sin\theta_1 \tag{34}$$

as $\theta_1 \to \pi n/2$ and $n = 1, 3, 5, \ldots$, or if the equation

$$\alpha = \frac{M}{2A}\sqrt{c_x^2 - \left(\frac{N}{BM}\right)^2} \tag{35}$$

is satisfied, where

$$\rho_* = \left(\frac{N}{BM}\right)^2 + \left(\frac{2\alpha A}{M}\right)^2, \quad \theta_1 = \operatorname{arc\,tg}\frac{N}{2\alpha AB}.$$

Analysis of the latter relationships shows that the most probable condition for the development of the stable waves under consideration on the surface of the film is

$$1 - \frac{\rho_1}{\rho_2} \approx \frac{1}{\rho_2 g}\left(\frac{\partial P}{\partial x} - 2\tau_0\right), \tag{36}$$

where the waves may have different lengths, velocities, and directions of propagation. If in (34) we pass to laminar flow and allow for the linearization of the original Eq. (27), it is not hard to derive the following Semenov formula [4]

$$\alpha = \mathrm{const}\, c_x \left(\frac{1}{\sigma h}\right)^{\frac{1}{2}}. \tag{37}$$

On the basis of (28), (29) and the solution to (31), we obtain the following expression for the amplitude function

$$\theta_1 = \frac{1}{\alpha}\left[\mathrm{const}\,\mathrm{sh}\,\alpha y + \int\limits_0^y P_*(L)\, e^{-\frac{z-a}{2}\sqrt{b_1}}\,\mathrm{sh}\,\alpha(y-z)\,dz\right]. \tag{38}$$

Thus in order to determine the additional resistance per unit length of flow due to the waves we have

$$\Delta P = \frac{\alpha \rho_2}{2\pi}\int\limits_0^\delta \theta_1^2(y)\,dy. \tag{39}$$

Comparison of (38) and (39) shows that the additional resistance is proportional to the wavelength and the square of the height of the wave.

NOTATION

r, x, θ^*	are cylindrical coordinates;
v_l	is the longitudinal velocity of the particles of the l-th component ($l = 2$ for the liquid and $l = 1$ for the gas);
v_{rl}	is the radial component of the velocity of the corresponding particles;
P_l	is the pressure inside the l-th component;
g	is the gravitational acceleration;
ρ_l	is the mass density of the corresponding particles;
δ	is the film thickness;
$\Delta\delta$	is the change in film thickness at the entrance section;
ν_l^T	is the generalized viscosity;
a_l	are experimental constants;
L_l	are the corresponding turbulence scales;
T_{rxl}	is the generalized frictional stress tensor in the equation of motion along the r axis;
$R_2 = 1$	is the radius of the tube;
σ	is the surface tension;
$v_{2\delta}$	is the velocity of the liquid on the surface of the film;
v_{20}	is the velocity distribution of liquid across the film section at the entrance to the tube;
v_{10}	is the velocity of the gas at the entrance into the tube, regarded as known;
v_{2lam}	is the velocity of the liquid at the boundary of the laminar underlayer;
Q_1, Q_2	are the volume flows of the gas and liquid, respectively;
ν_l	is the kinematic viscosity of the corresponding component;
τ_0	is the frictional stress at the surface of the liquid;
τ_{lam}	is the frictional stress at the tube wall for laminar motion of the liquid;
ψ	is an experimental coefficient;
A, B, C	are constants determined by the kinematic flow of the liquid;
$\theta(y)$	is the amplitude function;
t	is the time;
α	is the wave number;
λ	is the damping decrement;
M, N	are constants;
c_x	is the wave velocity;
h	is the height of the wave;
z	is an auxiliary variable.

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